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EXISTENCE AND GLOBAL EXPONENTIAL STABILITY OF A PERIODIC SOLUTION OF A HOPFIELD-TYPE NEURAL NETWORK WITH DISTRIBUTED DELAYS AND IMPULSES

VALÉRY COVACHEV AND ZLATINKA COVACHEVA

ABSTRACT. For a class of Hopfield-type neural networks with finite distributed delays and impulses in an integral form a sufficient condition for the existence of a periodic solution is obtained by using the Contraction Mapping Principle. If the system has a periodic solution, sufficient conditions are obtained for its uniqueness and global exponential stability introducing a suitable Lyapunov functional.

1. INTRODUCTION

A neural network is a network that performs computational tasks such as associative memory, pattern recognition, optimisation, model identification, signal processing, etc. on a given pattern via interaction between a number of interconnected units characterized by simple functions. Over the past two decades neural networks have been widely studied since they have been successfully applied to various processing problems such as optimisation, image processing, associative memory and many other fields (see [4] and references given therein). Different types of applications depend on the dynamical behaviours of the neural networks.

From the mathematical point of view, an artificial neural network corresponds to a nonlinear transformation of some inputs into certain outputs. Many types of neural networks have been proposed and studied in the literature and the Hopfield-type network has become an important one due to its potential for applications in various fields of daily life. The model proposed by Hopfield, also known as Hopfield's graded response neural network, is based on an analogue circuit consisting of capacitors, resistors and amplifiers.

In the present paper we find a sufficient condition for the existence of a periodic solution for a class of Hopfield-type neural networks with finite distributed

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delays and impulses in an integral form by using the Contraction Mapping Principle. If the system has a periodic solution, we obtain sufficient conditions for its uniqueness and global exponential stability introducing a suitable Lyapunov functional.

2. Statement of the problem. Preliminaries

We consider a class of Hopfield neural networks with integral impulsive conditions and finite distributed delays, which are formulated in the form of a system of impulsive delay differential equations

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -a_i(t)x_i(t) \\ &+ \sum_{j=1}^{m} b_{ij} f_j\left(\int_0^{\omega} K_{ij}(s)x_j(t-s)\,ds\right) + I_i(t), \qquad t \neq t_k, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \Delta x_i(t_k) &= -\alpha_{ik}x_i(t_k) \\ &+ \sum_{j=1}^{m} B_{ijk}\Phi_j\left(\int_{t_{k-1}}^{t_k} c_{ijk}(s)x_j(s)\,ds\right) + \gamma_{ik}, \quad k \in \mathbb{Z}, \quad i = \overline{1,m}, \end{aligned} \tag{2.2}$$

where m is the number of neurons in the network, $x_i(t)$ is the state of the i-th neuron at time t, $a_i(t) > 0$ is the rate at which the i-th neuron resets its state when isolated from the system; b_{ij} is the synaptic connection weight from the j-th neuron to the i-th one, $f_j(\cdot)$ are signal transmission functions of the j-th neuron; $\omega > 0$ is the maximum transmission delay from one neuron to another, $K_{ij}(\cdot)$ and $c_{ijk}(\cdot)$ are nonnegative integrable delay kernels; $I_i(t)$ is the external input to the i-th neuron; t_k ($k \in \mathbb{Z}$) are the instants of impulse effect which form a strictly increasing sequence, $\Delta x_i(t_k) := x_i(t_k+0) - x_i(t_k-0) \equiv x_i(t_k+0) - x_i(t_k)$; $\alpha_{ik} > 0$, γ_{ik} and B_{ijk} are constants; $\Phi_j(\cdot)$ are scalar functions.

Here and below for the solution $x(t)$ and other piecewise continuous functions we use the notation

$$x(t_k) = x(t_k - 0) = \lim_{t\to t_k,\, t<t_k} x(t), \quad x(t_k+0) = \lim_{t\to t_k,\, t>t_k} x(t).$$

We make the following assumptions:

[H1]: There exists a positive integer p such that $t_{k+p} = t_k + \omega$ for $k \in \mathbb{N}$. Moreover,

$$\begin{aligned} a_i(t+\omega) = a_i(t), \quad I_i(t+\omega) = I_i(t) \quad &\text{for} \quad t \in \mathbb{R} \quad \text{and} \quad i = \overline{1,m}, \\ \alpha_{i,k+p} = \alpha_{ik}, \quad \gamma_{i,k+p} = \gamma_{ik} \quad &\text{for} \quad k \in \mathbb{Z} \quad \text{and} \quad i = \overline{1,m}, \\ B_{ij,k+p} = B_{ijk} \quad &\text{for} \quad k \in \mathbb{Z} \quad \text{and} \quad i,j = \overline{1,m}, \end{aligned}$$

$$c_{ij,k+p}(s+\omega) = c_{ijk}(s) \quad \text{for} \quad k \in \mathbb{Z},\ s \in [t_{k-1}, t_k] \quad \text{and} \quad i,j = \overline{1,m}.$$

[H2]: The functions $a_i(t)$, $I_i(t)$ $(i = \overline{1,m})$ are piecewise continuous on $\mathbb{R}$ with possible discontinuities of the first kind at the points t_k $(k \in \mathbb{Z})$ where they are continuous from the left.

[H3]: There exist positive constants F_j, $\mathcal{F}_j$ such that

$$|f_j(x) - f_j(y)| \le F_j|x-y|, \quad |\Phi_j(x) - \Phi_j(y)| \le \mathcal{F}_j|x-y| \quad \text{for any} \quad x, y \in \mathbb{R}.$$

The Hopfield neural network (2.1) is similar to the bidirectional associative memory neural network considered in [6] and to Hopfield neural networks considered in our previous papers (see, for instance, [2]). The impulse conditions (2.2) are similar to those in our paper [1].

Without loss of generality we can assume that

$$0 = t_0 < t_1 < t_2 < \cdots < t_p = \omega. \tag{2.3}$$

Next we shall use an assertion which can be found in [3, Chapter 2, §7.1] using a little different notations.

Consider the linear impulsive system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + g(t), \quad t \ne t_k, \\ \Delta x(t_k) &= B_k x(t_k) + a_k, \quad k \in \mathbb{Z}, \end{aligned} \tag{2.4}$$

where x, $g : \mathbb{R} \to \mathbb{R}^m$, $a_k \in \mathbb{R}^m$, $A(t)$ and B_k are $(m \times m)$–matrices, $\{t_k\}_{k \in \mathbb{Z}}$ is a strictly increasing sequence such that $\lim\limits_{k \to \pm\infty} t_k = \pm\infty$.

Suppose that the following conditions hold.

[A1]: There exists $\omega > 0$ and $p \in \mathbb{N}$ such that $t_{k+p} = t_k + \omega$, $B_{k+p} = B_k$, $a_{k+p} = a_k$ $(k \in \mathbb{Z})$.

[A2]: The matrix-valued function $A(t)$ and the vector-valued function $g(t)$ are ω-periodic and piecewise continuous, with possible discontinuities of the first kind at the points t_k where they are continuous from the left.

[A3]: The matrices $E + B_k$, $k \in \mathbb{Z}$, are nonsingular (E — the unit $(m \times m)$–matrix).

Suppose, for the sake of definiteness, that the instants of impulse effect t_k satisfy (2.3). Together with (2.4) we consider the respective homogeneous system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t), \quad t \ne t_k, \\ \Delta x(t_k) &= B_k x(t_k), \quad k \in \mathbb{Z}. \end{aligned} \tag{2.5}$$

We assume that

[A4]: System (2.5) has a unique ω–periodic solution $x(t) \equiv 0$.

Let the $(m \times m)$–matrix $X(t)$ be the fundamental solution of (2.5) (*i.e.*, $X(0) = E$). Then condition [**A4**] implies that the matrix $E - X(\omega)$ is nonsingular. In such a case (see [5]) the nonhomogeneous system (2.4) has a unique ω–periodic solution $\varphi(t)$ given by the formula

$$\varphi(t) = \int_0^\omega G(t,\tau)g(\tau)\,d\tau + \sum_{k=0}^{p-1} G(t, t_k + 0)a_k,$$

where *Green's function of the periodic problem for the nonhomogeneous system corresponding to* (2.5) is defined by

$$G(t,\tau) = \begin{cases} X(t)(E - X(\omega))^{-1}X^{-1}(\tau), & 0 \le \tau < t \le \omega, \\ X(t+\omega)(E - X(\omega))^{-1}X^{-1}(\tau), & 0 \le t \le \tau \le \omega, \end{cases}$$

and extended as ω–periodic with respect to t, τ.

We apply the above assertion to a system of the form

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -a_i(t)x_i(t) + g_i(t), \qquad t \ne t_k, \\ \Delta x_i(t_k) &= -\alpha_{ik}x_i(t_k) + \Gamma_{ik}, \qquad k \in \mathbb{Z}, \quad i = \overline{1,m}. \end{aligned} \tag{2.6}$$

For this system condition [**A3**] takes the form:

[**H4**]**:** $\alpha_{ik} \ne 1$ for $i = \overline{1,m}$ and $k = \overline{0, p-1}$.

Further on, $X(t) = \operatorname{diag}(X_i(t),\ i = \overline{1,m})$, where

$$X_i(t) = \exp\left(-\int_0^t a_i(s)\,ds\right) \prod_{0 \le t_k < t} (1 - \alpha_{ik}).$$

In particular,

$$X_i(\omega) = \exp\left(-\int_0^\omega a_i(s)\,ds\right) \prod_{k=0}^{p-1} (1 - \alpha_{ik})$$

and condition [**A4**] takes the form:

[**H5**]**:** For $i = \overline{1,m}$ we have

$$\exp\left(-\int_0^\omega a_i(s)\,ds\right) \prod_{k=0}^{p-1} (1 - \alpha_{ik}) \ne 1.$$

If conditions [**H4**] and [**H5**] are satisfied, then Green's function of the periodic problem for the nonhomogeneous system corresponding to

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -a_i(t)x_i(t), \qquad t \ne t_k, \\ \Delta x_i(t_k) &= -\alpha_{ik}x_i(t_k), \qquad k \in \mathbb{Z}, \quad i = \overline{1,m}, \end{aligned}$$

is given by $G(t,\tau) = \operatorname{diag}(G_i(t,\tau),\ i = \overline{1,m})$, where

$$G_i(t,\tau) = \begin{cases} \dfrac{\exp\left(-\int_\tau^t a_i(s)\,ds\right) \prod\limits_{\tau \le t_k < t} (1-\alpha_{ik})}{1-\exp\left(-\int_0^\omega a_i(s)\,ds\right) \prod\limits_{k=0}^{p-1} (1-\alpha_{ik})}, & 0 \le \tau < t \le \omega, \\ \dfrac{\exp\left(-\int_\tau^{t+\omega} a_i(s)\,ds\right) \prod\limits_{\tau \le t_k < t+\omega} (1-\alpha_{ik})}{1-\exp\left(-\int_0^\omega a_i(s)\,ds\right) \prod\limits_{k=0}^{p-1} (1-\alpha_{ik})}, & 0 \le t \le \tau \le \omega, \end{cases} \tag{2.7}$$

and extended as ω–periodic with respect to t, τ.

The unique ω-periodic solution $\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_m(t))^T$ of system (2.6) is given by

$$\varphi_i(t) = \int_0^\omega G_i(t,\tau) g_i(\tau)\,d\tau + \sum_{k=0}^{p-1} G_i(t, t_k + 0)\Gamma_{ik}, \quad i = \overline{1,m}. \tag{2.8}$$

For $G_i(t,\tau)$ given by formula (2.7) we introduce the notation

$$G = \max_{i=\overline{1,m}} \sup_{t,\tau \in [0,\omega]} |G_i(t,\tau)|. \tag{2.9}$$

3. Existence of a unique periodic solution

Theorem 3.1. *Let system* (2.1), (2.2) *satisfy conditions* [**H1**] – [**H5**]. *If*

$$G \max_{i=\overline{1,m}} \left(\omega F_i \sum_{j=1}^m |b_{ji}| \int_0^\omega K_{ji}(s)\,ds \right. \\ \left. + \mathcal{F}_i \sum_{j=1}^m \sum_{k=0}^{p-1} |B_{jik}| \int_{t_{k-1}}^{t_k} c_{jik}(s)\,ds \right) < 1, \tag{3.1}$$

then system (2.1), (2.2) *has a unique ω-periodic solution.*

Proof. Let $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ be an ω-periodic solution of system (2.1), (2.2). Then $x(t)$ can be considered as an ω-periodic solution of a system of the form (2.6) with nonhomogeneities

$$\begin{aligned} g_i(t) &= \sum_{j=1}^m b_{ij} f_j \left(\int_0^\omega K_{ij}(s) x_j(t-s)\,ds \right) + I_i(t), \\ \Gamma_{ik} &= \sum_{j=1}^m B_{ijk} \Phi_j \left(\int_{t_{k-1}}^{t_k} c_{ijk}(s) x_j(s)\,ds \right) + \gamma_{ik}, \qquad k \in \mathbb{Z}, \quad i = \overline{1,m}, \end{aligned}$$

for which conditions [**A1**]–[**A4**] are satisfied. Thus $x(t)$ is given by equations of the form (2.8), that is, $x(t) = (x_1(t), x_2(t), \dots, x_m(t))^T$ is a solution of the system of integro-summary equations

$$\begin{aligned} x_i(t) &= \int_0^\omega G_i(t,\tau)\left[\sum_{j=1}^m b_{ij} f_j\left(\int_0^\omega K_{ij}(s)x_j(\tau - s)\,ds\right) + I_i(\tau)\right] d\tau \\ &+ \sum_{k=0}^{p-1} G_i(t, t_k + 0)\left[\sum_{j=1}^m B_{ijk}\Phi_j\left(\int_{t_{k-1}}^{t_k} c_{ijk}(s)x_j(s)\,ds\right) + \gamma_{ik}\right], \quad i = \overline{1,m}. \end{aligned} \tag{3.2}$$

System (3.2) can be written in an operator form as

$$x = \mathcal{P}x, \tag{3.3}$$

where the operator $\mathcal{P}$ is given by $\mathcal{P}x = (\mathcal{P}_1 x, \mathcal{P}_2 x, \dots, \mathcal{P}_m x)^T$,

$$\begin{aligned} (\mathcal{P}_i x)(t) &= \int_0^\omega G_i(t,\tau)\left[\sum_{j=1}^m b_{ij} f_j\left(\int_0^\omega K_{ij}(s)x_j(\tau - s)\,ds\right) + I_i(\tau)\right] d\tau \\ &+ \sum_{k=0}^{p-1} G_i(t, t_k + 0)\left[\sum_{j=1}^m B_{ijk}\Phi_j\left(\int_{t_{k-1}}^{t_k} c_{ijk}(s)x_j(s)\,ds\right) + \gamma_{ik}\right], \end{aligned}$$

$i = \overline{1,m}$. $x(t)$ is an ω-periodic solution of system (2.1), (2.2) if and only if x is a fixed point of the operator $\mathcal{P}$. We shall find a sufficient condition for $\mathcal{P}$ to act as a contraction in a suitable Banach space.

Denote by $\mathbb{X}$ the Banach space of functions $x : \mathbb{R} \to \mathbb{R}^m$, $x(t) = (x_1(t), x_2(t), \dots, x_m(t))^T$ which are ω-periodic and piecewise-continuous, with possible discontinuities of the first kind at $\{t_k\}_{k\in\mathbb{Z}}$ where they are continuous from the left, and equipped with the norm $\|x\| = \sum_{i=1}^m |x_i|$, where $|x_i| = \sup_{t\in(0,\omega]} |x_i(t)|$. It is easy to see that the operator $\mathcal{P}$ maps the space $\mathbb{X}$ into itself. It remains to show that under the assumptions of Theorem 3.1 $\mathcal{P}$ acts in $\mathbb{X}$ as a contraction.

Let $x, y \in \mathbb{X}$. Then we successively find

$$\begin{aligned}
&(\mathcal{P}_i x)(t) - (\mathcal{P}_i y)(t) \\
= &\int_0^\omega G_i(t,\tau) \sum_{j=1}^m b_{ij} \left[f_j \left(\int_0^\omega K_{ij}(s) x_j(\tau - s)\, ds \right) \right. \\
&\qquad \left. - f_j \left(\int_0^\omega K_{ij}(s) y_j(\tau - s)\, ds \right) \right] d\tau \\
+ &\sum_{k=0}^{p-1} G_i(t, t_k + 0) \sum_{j=1}^m B_{ijk} \left[\Phi_j \left(\int_{t_{k-1}}^{t_k} c_{ijk}(s) x_j(s)\, ds \right) \right. \\
&\qquad \left. - \Phi_j \left(\int_{t_{k-1}}^{t_k} c_{ijk}(s) y_j(s)\, ds \right) \right], \quad i = \overline{1,m};
\end{aligned}$$

$$\begin{aligned}
&|(\mathcal{P}_i x)(t) - (\mathcal{P}_i y)(t)| \\
\le\ &G \int_0^\omega \sum_{j=1}^m |b_{ij}| F_j \int_0^\omega K_{ij}(s) |x_j(\tau - s) - y_j(\tau - s)|\, ds\, d\tau \\
+\ &G \sum_{k=0}^{p-1} \sum_{j=1}^m |B_{ijk}| \mathcal{F}_j \int_{t_{k-1}}^{t_k} c_{ijk}(s) |x_j(s) - y_j(s)|\, ds, \quad i = \overline{1,m};
\end{aligned}$$

$$\begin{aligned}
|\mathcal{P}_i x - \mathcal{P}_i y| \ \le\ &G \sum_{j=1}^m \left(\omega F_j |b_{ij}| \int_0^\omega K_{ij}(s)\, ds \right. \\
+\ &\left. \mathcal{F}_j \sum_{k=0}^{p-1} |B_{ijk}| \int_{t_{k-1}}^{t_k} c_{ijk}(s)\, ds \right) |x_j - y_j|, \quad i = \overline{1,m};
\end{aligned}$$

$$\begin{aligned}
&\|\mathcal{P}x - \mathcal{P}y\| = \sum_{i=1}^m |\mathcal{P}_i x - \mathcal{P}_i y| \\
\le\ &G \sum_{i=1}^m \sum_{j=1}^m \left(\omega F_j |b_{ij}| \int_0^\omega K_{ij}(s)\, ds \right. \\
&\qquad \left. + \mathcal{F}_j \sum_{k=0}^{p-1} |B_{ijk}| \int_{t_{k-1}}^{t_k} c_{ijk}(s)\, ds \right) |x_j - y_j|
\end{aligned}$$

$$\begin{aligned}
&= G\sum_{i=1}^{m}\left(\omega F_i\sum_{j=1}^{m}|b_{ji}|\int_0^{\omega}K_{ji}(s)\,ds\right.\\
&\qquad\qquad\left.+\mathcal{F}_i\sum_{j=1}^{m}\sum_{k=0}^{p-1}|B_{jik}|\int_{t_{k-1}}^{t_k}c_{jik}(s)\,ds\right)|x_i-y_i|\\
&\le G\max_{i=\overline{1,m}}\left(\omega F_i\sum_{j=1}^{m}|b_{ji}|\int_0^{\omega}K_{ji}(s)\,ds\right.\\
&\qquad\qquad\left.+\mathcal{F}_i\sum_{j=1}^{m}\sum_{k=0}^{p-1}|B_{jik}|\int_{t_{k-1}}^{t_k}c_{jik}(s)\,ds\right)\sum_{i=1}^{m}|x_i-y_i|\\
&= G\max_{i=\overline{1,m}}\left(\omega F_i\sum_{j=1}^{m}|b_{ji}|\int_0^{\omega}K_{ji}(s)\,ds\right.\\
&\qquad\qquad\left.+\mathcal{F}_i\sum_{j=1}^{m}\sum_{k=0}^{p-1}|B_{jik}|\int_{t_{k-1}}^{t_k}c_{jik}(s)\,ds\right)\|x-y\|.
\end{aligned}$$

By virtue of inequality (3.1) we conclude that the operator $\mathcal{P}$ acts in the space $\mathbb{X}$ as a contraction. □

4. Global exponential stability of a periodic solution

In the sequel we shall use the following

Lemma 4.1. *Suppose that*

$$a_i(t) > F_i\sum_{j=1}^{m}|b_{ji}|\int_0^{\omega}K_{ji}(s)\,ds \tag{4.1}$$

for all $t\in(0,\omega]\cup\{t_k+0\}_{k=0}^{p-1}$ *and* $i=\overline{1,m}$.

Then there exists $\lambda^*>0$ *such that*

$$a_i(t)-\lambda-F_i\sum_{j=1}^{m}|b_{ji}|\int_0^{\omega}K_{ji}(s)e^{\lambda s}\,ds>0 \tag{4.2}$$

for all $\lambda\in[0,\lambda^*)$, $t\in(0,\omega]\cup\{t_k+0\}_{k=0}^{p-1}$ *and* $i=\overline{1,m}$.

Proof. Let us denote

$$\mathcal{H}_i(t,\lambda)=a_i(t)-\lambda-F_i\sum_{j=1}^{m}|b_{ji}|\int_0^{\omega}K_{ji}(s)e^{\lambda s}\,ds.$$

For each $t \in (0,\omega] \cup \{t_k + 0\}_{k=0}^{p-1}$ $\mathcal{H}_i(t,\lambda)$ is a continuous decreasing function of $\lambda \in [0,\infty)$ such that $\mathcal{H}_i(t,0) > 0$ by virtue of inequality (4.1) and $\lim_{\lambda\to+\infty} \mathcal{H}_i(t,\lambda) = -\infty$. Then it is easy to see that there exists $\lambda_i > 0$ such that $\mathcal{H}_i(t,\lambda) > 0$ for all $\lambda \in [0,\lambda_i)$ and $t \in (0,\omega] \cup \{t_k + 0\}_{k=0}^{p-1}$, while $\mathcal{H}_i(t,\lambda_i) = 0$ for some $t \in (0,\omega] \cup \{t_k + 0\}_{k=0}^{p-1}$. Now it suffices to choose $\lambda^* = \min_{i=\overline{1,m}} \lambda_i$. □

Our main result in the present section is the following

Theorem 4.2. *Let system* (2.1), (2.2) *have an ω-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_m^*(t))^T$ and satisfy the conditions* [**H1**]–[**H5**] *and inequalities* (4.1). *Then there exist constants $M > 1$ and $\lambda > 0$ such that any solution $x(t) = (x_1(t), x_2(t), \dots, x_m(t))^T$ of system* (2.1), (2.2) *defined at least for $t > -\omega$ satisfies the estimate*

$$\sum_{i=1}^{m} |x_i(t) - x_i^*(t)| \le Me^{-\lambda t} \prod_{k=1}^{i(0,t)} \Bigg\{ \max_{i=\overline{1,m}} |1 - \alpha_{ik}| + \max_{i=\overline{1,m}} \left(\mathcal{F}_i \sum_{j=1}^{m} |B_{jik}| \int_{t_{k-1}}^{t_k} e^{\lambda(t_k - s)} c_{jik}(s)\, ds \right) \Bigg\} \times \sum_{i=1}^{m} \sup_{s\in(-\omega,0]} |x_i(s) - x_i^*(s)| \quad \textit{for all } t > 0, \tag{4.3}$$

where $i(0,t) = \max\{k \in \{0\} \cup \mathbb{N} :\ t_k < t\}$ is the number of instants of impulse effect t_k in the interval $(0,t)$.

Proof. We have from (2.1) and condition [**H3**] that

$$D^+|x_i(t) - x_i^*(t)| \le -a_i(t)|x_i(t) - x_i^*(t)| + \sum_{j=1}^{m} |b_{ij}|\, F_j \int_0^{\omega} K_{ij}(s)|x_j(t-s) - x_j^*(t-s)|\, ds \tag{4.4}$$

for $i = \overline{1,m}$, $t > 0$, $t \ne t_k$, where $D^+f(t)$ denotes the upper right Dini derivative of a continuous function $f(t)$.

Let λ^* be the positive constant provided by Lemma 4.1. Next we define

$$z_i(t) = |x_i(t) - x_i^*(t)|e^{\lambda t}, \tag{4.5}$$

where $i = \overline{1,m}$, $t \in (-\omega,\infty)$ and $\lambda \in (0,\lambda^*)$. Then from (4.4) we derive

$$D^+z_i(t) \le -(a_i(t) - \lambda)z_i(t) + \sum_{j=1}^{m} |b_{ij}|F_j \int_0^{\omega} K_{ij}(s)e^{\lambda s} z_j(t-s)\, ds$$

for $t > 0,\ \ t \neq t_k$. We define a Lyapunov functional $V(\cdot)$ by
(4.6)

$$V(t) = \sum_{i=1}^{m} \left\{ z_i(t) + \sum_{j=1}^{m} |b_{ij}| F_j \int_0^{\omega} K_{ij}(s) e^{\lambda s} \left(\int_{t-s}^{t} z_j(\sigma)\, d\sigma \right) ds \right\}, \quad t > 0.$$

It is easily seen that $V(t) \geq 0$ for $t > 0$. We can now calculate the rate of change of $V(t)$ along the solutions of (2.1):

$$\begin{aligned} D^{+}V(t) &\leq \sum_{i=1}^{m} \left\{ -(a_i(t) - \lambda) z_i(t) + \sum_{j=1}^{m} |b_{ij}| F_j \int_0^{\omega} K_{ij}(s) e^{\lambda s}\, ds\, z_j(t) \right\} \\ &= -\sum_{i=1}^{m} \left\{ a_i(t) - \lambda - F_i \sum_{j=1}^{m} |b_{ji}| \int_0^{\omega} K_{ji}(s) e^{\lambda s}\, ds \right\} z_i(t) \leq 0 \end{aligned}$$

for $t > 0$, $t \neq t_k$, by virtue of (4.2). This implies that $V(t)$ is nonincreasing on each interval $(t_{k-1}, t_k]$, $k \in \mathbb{N}$, thus

$$V(t) \leq V(t_{k-1} + 0) \quad \text{for} \quad t \in (t_{k-1}, t_k],\ k \in \mathbb{N}. \tag{4.7}$$

In particular,

$$V(t_k) \leq V(t_{k-1} + 0), \qquad k \in \mathbb{N}. \tag{4.8}$$

Further on, making use of the equalities (2.2), for an arbitrary moment of impulse effect $t_k,\ k \in \mathbb{N}$, we successively find

$$\Delta(x_i - x_i^*)(t_k) = -\alpha_{ik}(x_i(t_k) - x_i^*(t_k))$$
$$+ \sum_{j=1}^{m} B_{ijk} \left\{ \Phi_j \left(\int_{t_{k-1}}^{t_k} c_{ijk}(s) x_j(s)\, ds \right) - \Phi_j \left(\int_{t_{k-1}}^{t_k} c_{ijk}(s) x_j^*(s)\, ds \right) \right\},$$

$$\begin{aligned} |x_i(t_k + 0) - x_i^*(t_k + 0)| &\leq |1 - \alpha_{ik}|\, |x_i(t_k) - x_i^*(t_k)| \\ &+ \sum_{j=1}^{m} |B_{ijk}| \mathcal{F}_j \int_{t_{k-1}}^{t_k} c_{ijk}(s)\, |x_j(s) - x_j^*(s)|\, ds, \end{aligned}$$

$$z_i(t_k + 0) \leq |1 - \alpha_{ik}| z_i(t_k)$$
$$+ \sum_{j=1}^{m} |B_{ijk}| \mathcal{F}_j \int_{t_{k-1}}^{t_k} e^{\lambda(t_k - s)} c_{ijk}(s) z_j(s)\, ds, \quad i = \overline{1, m}.$$

Making use of (4.6), (4.7) and (4.8), we obtain

$$\begin{aligned} V(t_k+0) &\le \max_{i=\overline{1,m}} |1-\alpha_{ik}| V(t_k) \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{m} |B_{jik}| \mathcal{F}_i \int_{t_{k-1}}^{t_k} e^{\lambda(t_k-s)} c_{jik}(s) z_i(s)\, ds \\ &\le \max_{i=\overline{1,m}} |1-\alpha_{ik}| V(t_k) \\ &+ \int_{t_{k-1}}^{t_k} e^{\lambda(t_k-s)} \max_{i=\overline{1,m}} \left(\mathcal{F}_i \sum_{j=1}^{m} |B_{jik}| c_{jik}(s) \right) V(s)\, ds \\ &\le \left\{ \max_{i=\overline{1,m}} |1-\alpha_{ik}| \right. \\ &+ \left. \max_{i=\overline{1,m}} \left(\mathcal{F}_i \sum_{j=1}^{m} |B_{jik}| \int_{t_{k-1}}^{t_k} e^{\lambda(t_k-s)} c_{jik}(s)\, ds \right) \right\} V(t_{k-1}+0) \end{aligned}$$

and

$$\begin{aligned} V(t) \le \prod_{k=1}^{i(0,t)} &\left\{ \max_{i=\overline{1,m}} |1-\alpha_{ik}| \right. \\ &+ \left. \max_{i=\overline{1,m}} \left(\mathcal{F}_i \sum_{j=1}^{m} |B_{jik}| \int_{t_{k-1}}^{t_k} e^{\lambda(t_k-s)} c_{jik}(s)\, ds \right) \right\} V(+0) \end{aligned} \tag{4.9}$$

for all $t>0$. Next, from (4.6) and (2.2) we find

$$\begin{aligned} V(+0) &= \sum_{i=1}^{m} \left\{ z_i(+0) + \sum_{j=1}^{m} |b_{ij}| F_j \int_0^{\omega} K_{ij}(s) e^{\lambda s} \left(\int_{-s}^{0} z_j(\sigma)\, d\sigma \right) ds \right\} \\ &\le \sum_{i=1}^{m} \left\{ |1-\alpha_{i0}| z_i(0) + \mathcal{F}_i \sum_{j=1}^{m} |B_{ji0}| \int_{t_{-1}}^{0} c_{ji0}(s) \left(e^{-\lambda s} z_i(s) \right)\, ds \right. \\ &\quad \left. + F_i \sum_{j=1}^{m} |b_{ji}| \int_0^{\omega} K_{ji}(s) e^{\lambda s} \left(\int_{-s}^{0} e^{\lambda\sigma} \left(e^{-\lambda\sigma} z_i(\sigma) \right) d\sigma \right) ds \right\} \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{m}\Bigg\{ |1-\alpha_{i0}|\,|x_i(0)-x_i^*(0)| \\
&+ \mathcal{F}_i \sum_{j=1}^{m} |B_{ji0}| \int_{t_{-1}}^{0} c_{ji0}(s)\,ds \sup_{s\in(t_{-1},0)} |x_i(s)-x_i^*(s)| \\
&+ F_i \sum_{j=1}^{m} |b_{ji}| \int_{0}^{\omega} K_{ji}(s)\frac{e^{\lambda s}-1}{\lambda}\,ds \sup_{s\in(-\omega,0)} |x_i(s)-x_i^*(s)| \Bigg\} \\
&\le \Bigg\{ \max_{i=\overline{1,m}} |1-\alpha_{i0}| + \max_{i=\overline{1,m}} \left(F_i \sum_{j=1}^{m} |b_{ji}| \int_{0}^{\omega} K_{ji}(s)\frac{e^{\lambda s}-1}{\lambda}\,ds \right) \\
&+ \max_{i=\overline{1,m}} \left(\mathcal{F}_i \sum_{j=1}^{m} |B_{ji0}| \int_{t_{-1}}^{0} c_{ji0}(s)\,ds \right) \Bigg\} \sum_{i=1}^{m} \sup_{s\in(-\omega,0]} |x_i(s)-x_i^*(s)|,
\end{aligned}
$$

that is,

$$
V(+0) \le M \sum_{i=1}^{m} \sup_{s\in(-\omega,0]} |x_i(s)-x_i^*(s)| \tag{4.10}
$$

with

$$
\begin{aligned}
M = \max_{i=\overline{1,m}} |1-\alpha_{i0}| \quad &+ \quad \max_{i=\overline{1,m}} \left(F_i \sum_{j=1}^{m} |b_{ji}| \int_{0}^{\omega} K_{ji}(s)\frac{e^{\lambda s}-1}{\lambda}\,ds \right) \\
&+ \quad \max_{i=\overline{1,m}} \left(\mathcal{F}_i \sum_{j=1}^{m} |B_{ji0}| \int_{t_{-1}}^{0} c_{ji0}(s)\,ds \right).
\end{aligned}
$$

Finally, the inequality

$$
\sum_{i=1}^{m} |x_i(t)-x_i^*(t) = e^{-\lambda t} \sum_{i=1}^{m} z_i(t) \le e^{-\lambda t} V(t)
$$

combined with (4.9) and (4.10) yields (4.3). □

Definition 4.3. The periodic solution $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_m^*(t))^T$ of system (2.1), (2.2) is said to be *globally exponentially stable* (with Lyapunov exponent λ) if there exist constants $\lambda > 0$ and $M \ge 1$ and any solution $x(t) = (x_1(t), x_2(t), \ldots, x_m(t))^T$ of system (2.1), (2.2) is defined for all $t > 0$ and we have

$$\sum_{i=1}^{m} |x_i(t) - x_i^*(t)| \leq Me^{-\lambda t} \sum_{i=1}^{m} \sup_{s\in(-\omega,0]} |x_i(s) - x_i^*(s)| \quad \text{for all} \quad t \geq 0. \tag{4.11}$$

For two sets of additional assumptions on the impulse effects we will show that inequality (4.3) implies global exponential stability of the periodic solution $x^*(t)$ of the impulsive system (2.1), (2.2).

Corollary 4.4. *Let all conditions of Theorem 4.2 hold. Let there exist* $\lambda \in (0, \lambda^*)$ *such that*

$$\max_{i=\overline{1,m}} |1 - \alpha_{ik}| + \max_{i=\overline{1,m}} \left(\mathcal{F}_i \sum_{j=1}^{m} |B_{jik}| \int_{t_{k-1}}^{t_k} e^{\lambda(t_k - s)} c_{jik}(s)\, ds \right) \leq 1 \tag{4.12}$$

for $k = \overline{0, p-1}$*. Then the periodic solution* $x^*(t)$ *of the impulsive system* (2.1), (2.2) *is globally exponentially stable with Lyapunov exponent* λ*.*

Because of the periodicity inequality (4.12) holds for all $k \in \mathbb{N}$. The proof of the corollary is obvious. The global exponential stability is provided by the rather small magnitudes of the impulse effects. Further we will show that we may have global exponential stability for quite large magnitudes of the impulse effects provided that these do not occur too often.

Corollary 4.5. *Let all conditions of Theorem 4.2 hold. Let there exist positive constants* $\lambda \in (0, \lambda^*)$ *and* B *satisfying the inequalities*

$$\max_{i=\overline{1,m}} |1 - \alpha_{ik}| + \max_{i=\overline{1,m}} \left(\mathcal{F}_i \sum_{j=1}^{m} |B_{jik}| \int_{t_{k-1}}^{t_k} e^{\lambda(t_k - s)} c_{jik}(s)\, ds \right) \leq B \tag{4.13}$$

for $k = \overline{0, p-1}$*, and* $\frac{p}{\omega} \ln B < \lambda$*. Then for any* $\tilde{\lambda} \in (0, \lambda - \frac{p}{\omega} \ln B)$ *the periodic solution* $x^*(t)$ *of the impulsive system* (2.1), (2.2) *is globally exponentially stable with Lyapunov exponent* $\tilde{\lambda}$*.*

Proof. Inequalities (4.3) and (4.13) yield

$$\sum_{i=1}^{m} |x_i(t) - x_i^*(t)| \leq Me^{-\lambda t} B^{i(0,t)} \sum_{i=1}^{m} \sup_{s\in(-\omega,0]} |x_i(s) - x_i^*(s)| \quad \text{for all} \quad t > 0.$$

Condition [**H1**] (the relation between the periods ω and p) implies that

$$\lim_{t\to\infty} \frac{i(0,t)}{t} = \frac{p}{\omega},$$

that is, for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that the inequality

$$\frac{i(0,t)}{t} \leq \frac{p}{\omega} + \varepsilon$$

is satisfied for all $t \geq T$. For such t we have $i(0,t) \leq \left(\frac{p}{\omega} + \varepsilon\right) t$ and

$$\sum_{i=1}^{m} |x_i(t) - x_i^*(t)| \leq M e^{-\left(\lambda - \left(\frac{p}{\omega} + \varepsilon\right) \ln B\right) t} \sum_{i=1}^{m} \sup_{s \in (-\omega, 0]} |x_i(s) - x_i^*(s)|.$$

It suffices to choose $\varepsilon > 0$ such that $\left(\frac{p}{\omega} + \varepsilon\right) \ln B < \lambda$, and denote

$$\tilde{\lambda} = \lambda - \left(\frac{p}{\omega} + \varepsilon\right) \ln B.$$

Then inequality (4.11) will be satisfied with $\tilde{\lambda}$ instead of λ and a possibly bigger constant M. □

5. Conclusion

In the present paper we found a sufficient condition for the existence of a periodic solution for a class of Hopfield-type neural networks with finite distributed delays and impulses in an integral form by using the Contraction Mapping Principle. For a system with a periodic solution, we obtained sufficient conditions for its uniqueness and global exponential stability introducing a suitable Lyapunov functional.

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